Analysis IA
Summer Assignment

Power to the Triangle:

Let $T = T(a, b, c)$ be ΔABC whose sides $BC = a$, $CA = b$, and $AB = c$ satisfy both $0 < a \le b \le c$ and $a + b > c$.

Define $T^2 = (a^2, b^2, c^2)$ to be the square of triangle T, provided that such a triangle exists.

Define $\sqrt{T} = T\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ to be the square root of triangle T, provided that such a triangle exists.

Use $K(T)$ for the area of triangle T and $P(T)$ for the perimeter of triangle T.

Results from a preceding problem may be used in a succeeding problem but not vice versa.
You may use Fermat's Last Theorem as a reason in any proof.

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1. a) Show that the square of an equilateral triangle is equilateral.

   b) Prove that the square of a right triangle does not exist.

2. Compute all x for which the area of $T = T(1, 1, x)$ is the same as the area of T^2.

3. a) Prove that T^2 exists if and only if T is acute.

   b) Prove that $\sqrt{T}$ exists for all T.

4. a) Prove that all the angles of $\sqrt{T}$ are acute.

   b) Prove that $\cos C \le \frac{1}{2}$ with equality if and only if T is equilateral.

5. Prove that the largest angle of T is at least as close to 60° as the largest angle of T^2.

Let $X_n = T(n-1, n, n+1)$ for n a real number.

6. a) Determine with proof, all n for which $\left(X_n\right)^2$, i.e., the square of X_n, is a triangle.

   b) Note that $X_{11} = T(10,11,12)$ can be squared producing triangle $T(100, 121, 144)$ and that this can be squared again producing triangle $T(10000, 14641, 20736)$. The latter, however, cannot be squared. Now for an integer p, there exists a real number k, $p \leq k \leq p+1$, such that for all $n \geq k$, X_n can be squared at least twice. Find p, justify your answer.

7. a) Provide, with reasons, a triangle which can be squared at least 2001 times and at most 2001 times.

   b) Let $N(T)$ be the number of times a triangle T can be squared. For example, $N(X_{11}) = 2$. Determine, with proof, all triangles T for which $N(T) = \infty$. Specify side relationships and angle restrictions.

8. a) Prove that if $n = \sqrt{a} + \sqrt{b} + \sqrt{c}$ for integers n, a, b, and c, then a, b, and c must be perfect squares.

   b) Using (a) or otherwise, prove that if $T = T(a, b, c)$ is a right triangle with integer sides, then the perimeter of $\sqrt{T}$ cannot be an integer.

9. Prove that there exists a right triangle T with the same perimeter as $\sqrt{T}$.

10. Define the reciprocal of $T(a, b, c)$ to be $T^{-1} = T\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. When T^{-1} is itself a triangle, define T to be invertible. Show that if T is invertible, then $a > \left(\frac{3-\sqrt{5}}{2}\right)c$.

A set of positive integers $\{x_1, x_2, \ldots\}$ is called a *Fibonacci set* if $x_1 < x_2$ and $x_n = x_{n-1} + x_{n-2}$ for all $n > 2$.

We say that $\{a, b\}$ is a subset of a Fibonacci set F if a and b are distinct but not necessarily consecutive members of F. Notice that any two-member set $\{a, b\}$ of positive integers is a subset of at least one Fibonacci set. Since any Fibonacci set is determined by specifying its two smallest members, list just the first two members. Thus, write $\{1, 2, \ldots\}$ for $\{1, 2, 3, 5, 8, 13, \ldots\}$.

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1. a) Give an example of a set of positive integers $\{a, b\}$ that is a subset of only one Fibonacci set.

   b) For how many Fibonacci sets $\{x_1, x_2, \ldots\}$ is it true that $x_3 = 2003$?

2. a) Find all Fibonacci sets that have $\{8, 144\}$ as a subset. Specify each one by listing its two smallest members.

   b) Let a and b be members of a Fibonacci set F with $0 < a < b$. Prove that a and b are not the smallest members of F if and only if $b \neq 2a$.

3. Prove that no Fibonacci set has $\{60, 117, 174\}$ as a subset.

4. Compute the number of Fibonacci sets where $x_n = 2003$ for $n \geq 2$.

5. Prove that any Fibonacci set has infinitely many subsets of the form $\{a, a + d, a + 2d\}$.

6. Prove that no Fibonacci set has a subset of the form $\{a, a + d, a + 2d, a + 3d\}$.

A set of positive integers $\{x_1, x_2, \ldots\}$ is called a *linear set* if there is a positive integer d for which $x_n = x_{n-1} + d$ for all $n > 1$.

7. a) Prove that the Fibonacci set $\{7, 11, \ldots\}$ and the linear set $\{8, 23, \ldots\}$ are disjoint.

   b) Is it possible for a Fibonacci set and a linear set to have members in common, but only *finitely many*? Prove your answer.

8. a) Prove that two Fibonacci sets can have exactly one member in common.

   b) Prove that two Fibonacci sets can have exactly two members in common.

9. a) Prove that whenever two Fibonacci sets have more than two members in common, then the set of common members is itself a Fibonacci set.

   b) Prove: Given any Fibonacci sets $F_1, F_2, \ldots, F_n$, there is a positive integer that does not belong to any of them.

10. Can the set of positive integers be partitioned into Fibonacci sets? This means finding infinitely many pair-wise disjoint Fibonacci sets whose union contains every positive integer. Prove your answer.